Some general theorems concerning the finite motion of a shallow rotating liquid lying on a paraboloid

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The motion of a bounded shallow liquid, initially of arbitrary shape, in an arbitrary state of motion and lying on a paraboloid of revolution (including a level surface as a special case) can always be separated into three parts:

(i) the motion of the centre of gravity, which is entirely independent of the other motions and is governed by a pair of simple ordinary linear differential equations;

(ii) an isotropic two-dimensional dilatation and rotation which are also governed by a simple linear differential equation;

(iii) the motions that remain after removal of the velocity fields associated with the preceding motions; these will be called additional motions.

The additional motions exert a 'pressure', determined by their total energy, which tends to increase the spread of the liquid. If the spread does increase then the additional motions lose energy which then appears as energy associated with the dilatation.

The effect of the dilatation and rotation on the additional motions can be described by transformation into a co-ordinate system that rotates and dilates with the liquid. In these co-ordinates, with a properly adjusted time scale, the additional motions satisfy equations that are isomorphic with the original equations of motion; however, the liquid now appears to be lying on a parabola that is *always concave upwards*.

1. Introduction

In the following analysis we are concerned with motions of a liquid that is 'shallow' in the sense that vertical accelerations can be neglected. On the other hand the horizontal scale may be large enough for coriolis forces due to the rotation of the earth to be important. A bounded mass of liquid is considered and friction neglected, so that energy can be neither radiated away by gravity waves nor destroyed. The total energy is therefore constant and this constant plays an important part in the later developments of the theory. The liquid is assumed to lie on a paraboloid that is not necessarily concave upwards and the theory describes not only oscillatory motions but also a general collapse under gravity.

The basic theory is divided into two sections; the first and lesser section being concerned with 'displacement', i.e. motion of the centre of gravity of the liquid; and the second being concerned with 'distension', a term which is here used to signify a combination of rotation and two-dimensional dilatation. In each case a general theorem is first proved which shows how the displacement or distension can be determined regardless of an arbitrary field of additional motions. The effect of the displacement or distension on the other motions is then completely described by a transformation of co-ordinates. It is found that the displacement is independent of other motions, whereas there is a non-linear interaction involving transfer of energy between the distension and the other motions. The latter case is of special interest because the non-linear interaction is described completely and exactly; furthermore, it is not an interaction between two particular solutions but an interaction between a particular solution and all other shallow water motions.

Previous work has shown that the *linearized* shallow-water equations have particularly simple solutions when the motion occurs on a paraboloidal surface; see, for instance, Proudman (1925), Goldsbrough (1930) and Lamb (1932, §§ 193 and 212). The results presented here show that this simplicity extends in some degree to the *non-linear* equations. It has also been shown (Ball 1962) that the paraboloid is the only surface on which a shallow liquid can move in the simple way described hereafter.

There is a difficulty, concerning the validity of the shallow water approximation, that is tacitly ignored throughout the remainder of the paper. It is usually a simple matter to justify the shallow-water approximation *a posteriori*, once a particular solution has been determined (see Lamb 1932, § 172). In the present case, where the main concern is with general theorems rather than particular solutions, this is not possible (except for the two particular solutions presented). It would be very useful if one could either impose some limitation on the initial conditions to ensure the continued validity of the approximation or deduce from given initial conditions a time interval during which the approximation should remain valid.

Usually, after a time, solutions of the shallow-water equations cease to be single valued and so become physically meaningless (though this situation can be partly remedied by introducing discontinuities into the flow). However, before this stage is reached, the horizontal gradients become indefinitely large and the shallow water approximation ceases to be applicable somewhere in the flow. It seems likely, that by ensuring the validity of the shallow water approximation, we would also ensure the existence of meaningful solutions.

2. Displacement theory

The 'shallow water' or 'tidal' form of the non-linear equations of liquid motion is assumed throughout the following analysis and no further approximations are made. These equations are

$$\frac{Du}{Dt}+g\frac{\partial}{\partial x}(h+Z) = fv, \tag{1}$$

$$\frac{Dv}{Dt}+g\frac{\partial}{\partial y}(h+Z) = -fu, \tag{2}$$

$$\frac{Dh}{Dt}+h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0, \tag{3}$$

where h is the depth of the liquid and Z is the height of the base of the liquid above some fixed level. The height of the liquid surface above this level is equal to $h+Z$ and the slope of this surface determines the horizontal pressure gradient within the liquid. The velocities u, v are independent of z, and $f = 2\Omega \sin\phi$, the coriolis parameter, where Ω is the angular velocity of the earth and ϕ the latitude.

We now investigate the movement of the centre of gravity of a finite volume of liquid, with free boundaries, lying on a paraboloidal surface; so that the function Z has the form $\frac{1}{2}(\alpha x^2+\beta y^2)$. Before developing the main theorem we mention the following lemma

$$\frac{d}{dt}\int hq\,dS = \int h\frac{Dq}{Dt}\,dS, \tag{4}$$

where q is any quantity, dS is an element of area and the integration extends over the area occupied by a given volume of liquid, the boundaries moving with the liquid. The validity of equation (4) is evident since by proper interpretation of dS we have $D(h\,dS)/Dt = 0$.

Let the co-ordinates of the centre of gravity of a given volume of liquid be X and Y, then

$$QX = \int hx\,dS \quad \text{and} \quad QY = \int hy\,dS,$$

where $Q = \int h\,dS$, the constant total volume of the liquid. Whence by differentiating with respect to time and using equation (4) we find

$$Q\frac{dX}{dt} = \int h\frac{Dx}{Dt}\,dS = \int hu\,dS,$$

and

$$Q\frac{d^2X}{dt^2} = \int h\frac{Du}{Dt}\,dS,$$

and corresponding relationships for Y. Now the equation of motion in the x direction is

$$\frac{Du}{Dt}+g\left(\frac{\partial h}{\partial x}+\alpha x\right) = fv.$$

On multiplying through by h and integrating each term over the area occupied by the liquid, we find

$$Q\frac{d^2X}{dt^2}+Qg\alpha X+g\int\frac{\partial}{\partial x}(\tfrac{1}{2}h^2)\,dS = fQ\frac{dY}{dt}.$$

Because we are considering a volume of liquid with 'free' boundaries, where h is zero, the third term on the left-hand side vanishes giving

$$\frac{d^2X}{dt^2}+g\alpha X = f\frac{dY}{dt}, \tag{5}$$

and from the equation of motion in the y direction

$$\frac{d^2Y}{dt^2}+g\beta Y = -f\frac{dX}{dt}. \tag{6}$$

Thus the motion of the centre of gravity is determined uniquely if its initial velocity and position are known. In other words *the displacement of the centre of* *gravity of the liquid is independent of the motion that occurs within the liquid relative to the centre of gravity.*

We now show that the converse of the preceding statement is also true. The total motion satisfies the shallow-water equations (1) to (3). Let us put

$$u' = u - dX/dt,$$
$$v' = v - dY/dt,$$
$$x' = x - X,$$

and $$y' = y - Y,$$

so that u' and v' are the velocities relative to the centre of gravity and x' and y' are the coordinates measured from the centre of gravity. With these variables the equations transform to

$$\frac{Du'}{Dt} + g\left(\frac{\partial h}{\partial x'} + \alpha x'\right) = fv', \tag{7}$$

$$\frac{Dv'}{Dt} + g\left(\frac{\partial h}{\partial y'} + \beta y'\right) = -fu', \tag{8}$$

$$\frac{Dh}{Dt} + h\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) = 0, \tag{9}$$

Therefore the equations governing the motion of the liquid relative to its centre of gravity have exactly the same form as the original equations of motion.

We have now shown that the motion of the centre of gravity of a finite volume of liquid with free boundaries, lying on a paraboloidal surface, is independent of the motions relative to the centre of gravity and vice versa. This independence is rather different from that of linear solutions. Here the centre of gravity is displaced and other motions occurring in the liquid are displaced with it; these other motions have no effect whatever on the displacement and the displacement has no effect whatever on the other motions. The liquid may be in geostrophic or cyclostrophic equilibrium, it may be collapsing under its own weight or contorting itself into a new shape under the influence of coriolis forces; whatever it is doing the motion of its centre of gravity is unaffected. In particular the frequency of oscillatory displacement motions is independent of the rotation of the liquid (see Miles & Ball 1963).

There are two special cases of particular interest. First, let us suppose that $u' = v' = 0$, i.e. there is no (horizontal) motion relative to the centre of gravity and the liquid undergoes a quasi-rigid displacement. The corresponding solution for h, from equations (7) to (9), is then

$$h_0 = h_c - \tfrac{1}{2}(\alpha x'^2 + \beta y'^2),$$

where h_c is the constant central depth of the liquid. If the liquid is finite then in this special case α and β are necessarily positive and we are concerned with a paraboloid that is concave upwards. Returning to the original variables, we find the following simple *exact* solution of the non-linear shallow water equations

$$u_0 = dX/dt, \tag{10}$$

$$v_0 = dY/dt, \tag{11}$$

$$h_0 = h_c - \tfrac{1}{2}[\alpha(x-X)^2 + \beta(y-Y)^2], \tag{12}$$

where X and Y satisfy the simple linear equations (5) and (6). Such a motion will be called a *pure displacement*.

We now show that a pure displacement has the minimum energy compatible with a given motion of the centre of gravity and given volume Q. Consider a liquid state u, v, h with known X, dX/dt, Y, dY/dt and Q, but otherwise arbitrary. Now

$$\int hu\,dS = Q\,dX/dt = \int hu_0\,dS,$$

and similarly for v, whence

$$\int hu'\,dS = \int hv'\,dS = 0,$$

and

$$\frac{1}{2}\int h(u^2+v^2)\,dS = \frac{1}{2}\int h[(u_0+u')^2+(v_0+v')^2]\,dS$$

$$= \frac{1}{2}\int h(u_0^2+v_0^2)\,dS + \frac{1}{2}\int h(u'^2+v'^2)\,dS.$$

Therefore the kinetic energy of the arbitrary motion is the sum of the kinetic energies of the corresponding displacement and of the motions relative to the centre of gravity; furthermore, the kinetic energy of the arbitrary motion is always greater than that of the displacement (unless $u' = v' = 0$ everywhere). This result is a special case of the general theorem proved, for instance, in Lamb's *Dynamics* (1923) § 46.

To show that the potential energy is also a minimum we put $h = h_0 + h'$ and then we have

$$\int h\,dS = Q = \int h_0\,dS \quad \text{whence} \quad \int h'\,dS = 0.$$

The integration extends over the whole area in which the integrand is not zero. We also have

$$\int hx\,dS = QX = \int h_0 x\,dS,$$

and a similar relationship for y, whence

$$\int h'x\,dS = \int h'y\,dS = 0.$$

The potential energy is given by

$$\tfrac{1}{2}g\int h(\alpha x^2+\beta y^2+h)\,dS = \tfrac{1}{2}g\int (h_0+h')(\alpha x^2+\beta y^2+h_0+h')\,dS$$

$$= \tfrac{1}{2}g\int h_0(\alpha x^2+\beta y^2+h_0)\,dS + \tfrac{1}{2}g\int h'^2\,dS$$

$$+\tfrac{1}{2}g\int h'(\alpha x^2+\beta y^2+2h_0)\,dS,$$

and by virtue of the preceding conditions and the definition of h_0 (equation (12)), the last term vanishes. Therefore the potential energy of the liquid is always greater than the potential energy of a pure displacement (unless $h = h_0$). We have shown that the pure displacement not only has the minimum total energy compatible with the given motion of the centre of gravity, but also has the minimum kinetic energy and the minimum potential energy.

Another simple property of a pure displacement concerns the shape of the free surface, $z = h+Z$. The quadratic terms in h are equal and opposite to those in Z so that $h+Z$ is a linear function of x and y. Thus the free surface is always plane, as is also the case for the corresponding modes in the linear theory (see Goldsbrough 1930).

As our second special case, consider the motion of the centre of gravity when the underlying surface is a paraboloid of revolution. Equations (5) and (6) have energy and angular momentum integrals (the constant density is omitted from these and similar results presented later)

$$E_c = \tfrac{1}{2}Q\left[\left(\frac{dX}{dt}\right)^2 + \left(\frac{dY}{dt}\right)^2 + \alpha g(X^2+Y^2)\right],$$

$$J_c = Q\left[X\frac{dY}{dt} - Y\frac{dX}{dt} + \tfrac{1}{2}f(X^2+Y^2)\right],$$

which express constancy of energy E_c (this is the energy of the motion and position of the centre of gravity, not the total energy) and absolute angular momentum J_c (this represents the sum of the apparent angular momentum of the centre of gravity about the origin and the angular momentum imparted by the earth's rotation). If we put $R^2 = X^2+Y^2$, then we find

$$\frac{d^2R^2}{dt^2} + (f^2+4\alpha g)\,R^2 = \frac{4E_c+2fJ_c}{Q}, \tag{13}$$

which shows that the motion is stable if $f^2+4\alpha g > 0$. These special cases are also discussed elsewhere (see Ball 1962).

3. Distension theory

Consider once more a finite volume of liquid, with free boundaries, lying on a paraboloid of revolution. It is convenient to use polar co-ordinates and the shallow water equations then take the form

$$\frac{Du}{Dt} + g\left(\frac{\partial h}{\partial r} + \alpha r\right) = \left(f+\frac{v}{r}\right)v, \tag{14}$$

$$\frac{Dv}{Dt} + \frac{g}{r}\frac{\partial h}{\partial \theta} = -\left(f+\frac{v}{r}\right)u, \tag{15}$$

$$\frac{Dh}{Dt} + \frac{h}{r}\left(\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial \theta}\right) = 0, \tag{16}$$

where u and v are now the radial and tangential velocities respectively and will retain this meaning throughout the remainder of the paper. The total energy of the liquid is

$$E = \int \tfrac{1}{2}h(u^2+v^2+gh+\alpha g r^2)\,dS, \tag{17}$$

and the total absolute angular momentum† about a vertical axis through the origin, is

$$J = \int hr(v+\tfrac{1}{2}fr)\,dS, \tag{18}$$

† The absolute angular momentum is the sum of the relative angular momentum, associated with the velocity relative to the earth, and the angular momentum imparted by the earth's rotation.

where the integration extends over the whole area covered by the liquid. *Both E and J are constants* (determined by the initial state) as one can readily verify by differentiation, using equations (4), (14) to (16) and the fact that h is zero on the boundary. For present purposes it is convenient to introduce a further constant, K, the 'absolute energy', where

$$K = E + \tfrac{1}{2}fJ$$
$$= \int \tfrac{1}{2}h[u^2 + (v + \tfrac{1}{2}fr)^2 + gh + \tfrac{1}{4}\nu^2 r^2]\, dS, \tag{19}$$

and
$$\nu^2 = f^2 + 4\alpha g. \tag{20}$$

The absolute energy is essentially positive if ν^2 is positive; which, as we shall see, is also the condition for stability of the system.

We now show that the moment of inertia of the liquid about a vertical axis through the origin satisfies a strikingly simple equation. This moment of inertia is defined as

$$I = \int hr^2\, dS,$$

whence, using equation (4), we find

$$\frac{dI}{dt} = 2\int hru\, dS,$$

$$\frac{d^2I}{dt^2} = 2\int h\left(u^2 + r\frac{Du}{Dt}\right) dS$$
$$= 2\int \left\{h[u^2 + (v + \tfrac{1}{2}fr)^2 + gh - \tfrac{1}{4}\nu^2 r^2] - \frac{g}{2r}\frac{\partial}{\partial r}(r^2h^2)\right\} dS.$$

The last term in the integrand vanishes because h is zero on the boundary, and so from equation (19)

$$\frac{d^2I}{dt^2} + \nu^2 I = 4K. \tag{21}$$

By multiplying through by dI/dt and integrating we obtain

$$\frac{1}{2}\left[\left(\frac{dI}{dt}\right)^2 + \nu^2 I^2\right] = 4KI - C, \tag{22}$$

where C is a constant of integration which, as we shall see subsequently, is necessarily positive.

These equations constitute the first important result in this section. They imply that *the value of the moment of inertia of the liquid about a vertical axis through the origin is determined uniquely at all times if its initial value and rate of change are known, together with the total absolute energy of the liquid.* Thus the behaviour of the moment of inertia is independent of the details of the motions present in the liquid mass.

Before proceeding to more general theory we notice that equation (21) is consistent with the results of the previous section. If we suppose that the moment of inertia about the centre of gravity is I_c, then $I = I_c + R^2Q$, and from equation (21)

$$\frac{d^2I_c}{dt^2} + \frac{Qd^2R^2}{dt^2} = 4K - \nu^2(I_c + R^2Q),$$

whence, eliminating R by using equation (13), we find

$$\frac{d^2I_c}{dt^2} = 4(K - K_c) - \nu^2 I_c,$$

where $K_c = E_c + \frac{1}{2}fJ_c$ is the absolute energy associated with the position and motion of the centre of gravity. Thus I_c is governed by an equation of exactly the same form as I, $K - K_c$ being the absolute energy of the motions relative to the centre of gravity.

We have shown that the changes in moment of inertia are independent of the details of the motion of the liquid and we now investigate the extent to which the detailed motions are independent of the changes in moment of inertia. In order to do this we must separate the velocity field into two parts, one that is directly associated with the changes in moment of inertia and one that is not. The simplest radial velocity field that can be so associated is that produced by a *uniform isotropic expansion* about the origin; the simplest tangential velocity field is that produced by a *uniform angular velocity*. This velocity field U, V is determined uniquely when I, dI/dt and J are known, viz.

$$U = \frac{r}{2I}\frac{dI}{dt} = \frac{r}{I}\int hru\,dS, \tag{23}$$

and

$$V = r\left(\frac{J}{I} - \frac{f}{2}\right) = \frac{r}{I}\int hrv\,dS. \tag{24}$$

For brevity this particular combination of rotation and two-dimensional dilatation will be called a *distension*. There are at least two reasons for claiming that this motion is the simplest compatible with a given I, dI/dt and J. First, any lack of uniformity or isotropy would have to be specified by some additional parameter, thus increasing the complexity, e.g. a pure displacement would not be determined uniquely by I, dI/dt and J, further information is required concerning the direction of the displacement. Secondly, a distension has the least kinetic energy compatible with the given conditions, as will be proved subsequently.

Let us return to the general case, where the liquid is in an arbitrary state of (shallow water) motion. The quantities I, dI/dt and J are readily determined at all times from the initial state and we can always define a distension by means of equations (23) and (24). Furthermore, the velocity field can then be regarded as the sum of the distension and an additional velocity field u', v', i.e. $u' = u - U$ and $v' = v - V$. We now refer the additional motions to a co-ordinate system that distends, i.e. expands and rotates, with the liquid (this procedure is analogous to taking co-ordinates that move with the centre of gravity of the liquid in the displacement theory). To preserve the property of incompressibility when using these horizontal co-ordinates there must be an appropriate vertical contraction accompanying a horizontal expansion; this suggests that h should also be transformed. In order to maintain compatibility of u'^2, v'^2 and gh we find transformations for u' and v'. Finally, comparison of the transformations for r and u' suggests the transformation for t so that a relation of the form $Dr/Dt = u$ will be retained. This reasoning, briefly outlined above, leads to the transformation

$$r = r^*(I/I_0)^{\frac{1}{2}}, \tag{25}$$

$$\theta = \theta^* + \int (J/I - \tfrac{1}{2}f)\,dt, \tag{26}$$

$$dt = dt^*(I/I_0), \tag{27}$$

for the independent variables, and

$$h = h^*(I_0/I), \tag{28}$$

$$u = u' + U = u^*(I_0/I)^{\frac{1}{2}} + \frac{r}{2I}\frac{dI}{dt}, \tag{29}$$

$$v = v' + V = v^*(I_0/I)^{\frac{1}{2}} + r(J/I - \tfrac{1}{2}f), \tag{30}$$

for the dependent variables. The new variables have been denoted by asterisks and I_0 is a constant (as yet unspecified) reference moment of inertia.

We now prove the important result that when the equations of motion and continuity are transformed, using these new variables, the resulting equations have exactly the same form as the original ones. To prove this we put

$$\begin{aligned} \frac{\partial}{\partial t} &\equiv \frac{\partial t^*}{\partial t}\frac{\partial}{\partial t^*} + \frac{\partial r^*}{\partial t}\frac{\partial}{\partial r^*} + \frac{\partial \theta^*}{\partial t}\frac{\partial}{\partial \theta^*} \\ &= (I_0/I)\frac{\partial}{\partial t^*} - \tfrac{1}{2}(I_0/I)^{\frac{1}{2}}\frac{r}{I}\frac{dI}{dt}\frac{\partial}{\partial r^*} - (J/I - \tfrac{1}{2}f)\frac{\partial}{\partial \theta^*}, \end{aligned}$$

$$\frac{\partial}{\partial r} \equiv (I_0/I)^{\frac{1}{2}}\frac{\partial}{\partial r^*},$$

$$\frac{\partial}{\partial \theta} \equiv \frac{\partial}{\partial \theta^*}.$$

So

$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} \equiv (I_0/I)\left(\frac{\partial}{\partial t^*} + u^*\frac{\partial}{\partial r^*} + \frac{v^*}{r^*}\frac{\partial}{\partial \theta^*}\right),$$

or

$$\frac{D}{Dt} \equiv (I_0/I)\frac{D^*}{D^*t^*}.$$

Using these results, we find, by straightforward transformation of the equations of motion and continuity (equations (14) to (16)), the following equations in the new variables

$$\frac{D^*u^*}{D^*t^*} + g\frac{\partial h^*}{\partial r^*} + \alpha^* g r^* = (f^* + v^*/r^*)\,v^*, \tag{31}$$

$$\frac{D^*v^*}{D^*t^*} + \frac{g}{r^*}\frac{\partial h^*}{\partial \theta^*} = -(f^* + v^*/r^*)\,u^*, \tag{32}$$

$$\frac{D^*h^*}{D^*t^*} + \frac{h^*}{r^*}\left(\frac{\partial}{\partial r^*}(r^*u^*) + \frac{\partial v^*}{\partial \theta^*}\right) = 0, \tag{33}$$

where the new 'coriolis parameter' is given by

$$f^* = 2J/I_0, \tag{34}$$

and the apparent shape of the paraboloid is determined by

$$\alpha^* g = (C - 2J^2)/2I_0^2. \tag{35}$$

The reduction of the equation of radial motion to the above form requires the use of equations (21) and (22) which govern the changes of moment of inertia. *The preceding equations are remarkable in that they have exactly the same form as the original equations of motion.* The motion described by these equations is, however, simpler than the original motion; primarily because there is no distension present and α^* is always positive. Thus

$$I^* = \int h^* r^{*2}\, dS^* = I_0, \quad \text{constant}, \tag{36}$$

and
$$J^* = \int h^* r^* (v^* + \tfrac{1}{2} f^* r^*)\, dS^* = J, \quad \text{constant},$$

so
$$\int h^* r^* u^*\, dS^* = \int h^* r^* v^*\, dS^* = 0. \tag{37}$$

Perhaps the most elegant way to prove that α^* is positive is by consideration of the quantities ν^*, E^* and K^*, defined in terms of the new variables in exactly the same way as ν, E and K were defined in terms of the old. Thus

$$\nu^{*2} = 4\alpha^* g + f^{*2}. \tag{38}$$

Furthermore, from the equation in the new variables which is analogous to equation (21) in the old, and remembering that $I^* = I_0$ (equation (36)), we find

$$\nu^{*2} I_0 = 4K^*.$$

Now by definition of K^* (see equation (19)) we have

$$E^* = K^* - \tfrac{1}{2} f^* J,$$

whence eliminating K^* and ν^{*2} and putting $f^* = 2J/I_0$ we obtain

$$\alpha^* g = E^*/I_0. \tag{39}$$

To complete the proof we use the definition of E^*

$$E^* = \frac{1}{2}\int h^* (u^{*2} + v^{*2} + gh^* + \alpha^* g r^{*2})\, dS^*,$$

whence from equation (39),

$$\tfrac{1}{2} E^* = \frac{1}{2}\int h^* (u^{*2} + v^{*2} + gh^*)\, dS^*, \tag{40}$$

and, because all the terms in the integrand are positive,

$$E^* > 0 \quad \text{and} \quad \alpha^* > 0.$$

The positiveness of α^* is of great importance. It means that the additional motions must be regarded as occurring on a paraboloid that is concave upwards and these motions are therefore stable. The *only* form of instability that can occur is an unstable *distension* the condition for which is $4\alpha g + f^2 \leqslant 0$. It should be noted, however, that additional motions, stable in the asterisked co-ordinates, may become unstable on transformation to real co-ordinates if $4\alpha g + f^2 \leqslant 0$.

Before proceeding to more general theory we investigate an important special case. Equations (31) to (33) have the simple equilibrium solution

$$u^* = v^* = 0,$$
$$h^* = H^* = h_0 - \tfrac{1}{2}\alpha^* r^{*2}, \tag{41}$$

where α^* now has the value $I_0^{-1}\int H^{*2}\,dS^*$. In terms of the known volume Q and the moment of inertia I_0, equation (41) takes the form

$$H^* = \frac{2Q^2}{3\pi I_0}\left(1 - \frac{Qr^{*2}}{3I_0}\right). \tag{42}$$

If we return to the original variables using the transformations (25) to (30) we find the following *exact* solution of the non-linear shallow-water equations

$$U = \frac{r}{2I}\frac{dI}{dt},$$
$$V = r\left(\frac{J}{I} - \frac{f}{2}\right),$$
$$H = \frac{2Q^2}{3\pi I}\left(1 - \frac{Qr^2}{3I}\right), \tag{43}$$

where I is a function of time given by a simple linear differential equation (equation (21)). This solution will be called a *pure distension*. This special case is also discussed elsewhere (Ball 1962; Miles & Ball 1963). The mean and extreme positions of the free surface in simple examples of a pure displacement and a pure distension are indicated in figure 1.

A pure distension has the minimum energy compatible with a given Q, I, dI/dt and J as will now be shown. Consider a liquid state u, v, h, subject to these limitations but otherwise arbitrary. Now

$$\frac{1}{2}\int h(u^2+v^2)\,dS = \frac{1}{2}\int h\left\{\left[U + u^*\left(\frac{I_0}{I}\right)^{\frac{1}{2}}\right]^2 + \left[V + u^*\left(\frac{I_0}{I}\right)^{\frac{1}{2}}\right]^2\right\} dS$$
$$= \frac{1}{2}\int H(U^2+V^2)\,dS + \frac{I_0}{2I}\int h^*(u^{*2}+v^{*2})\,dS^*.$$

The product terms vanish by virtue of equation (37). The last term is positive (unless $u = U$ and $v = V$) and the first term is the kinetic energy of a pure distension; therefore the kinetic energy of the arbitrary liquid state is always greater than that of the pure distension. Similarly the potential energy is given by

$$\tfrac{1}{2}g\int h(h+\alpha r^2)\,dS = \tfrac{1}{2}g\left\{\int h^2\,dS + \alpha I\right\}$$
$$= \tfrac{1}{2}g\left\{\int H^2\,dS + \alpha I + \int h'^2\,dS\right\},$$

where $h = H + h'$. The product terms vanish because constancy of Q implies $\int h'\,dS = 0$, and from invariance of I we have $\int h' r^2\,dS = 0$, so $\int Hh'\,dS = 0$. We see therefore that the potential energy of the arbitrary liquid shape always exceeds that of a pure distension (unless $h = H$). We have therefore proved, not only that a pure distension has the minimum energy compatible with given $Q, I, dI/dt$ and J, but also a minimum kinetic energy and a minimum potential energy. The minimum value of the energy is

$$E_m = \frac{1}{2}\int H(U^2 + V^2 + gH)\,dS + \tfrac{1}{2}\alpha g I$$
$$= \frac{1}{8I}\left\{\left(\frac{dI}{dt}\right)^2 + 4\alpha g I^2\right\} + \frac{(J - \frac{1}{2}fI)^2}{2I} + \frac{2Q^3 g}{9\pi I},$$

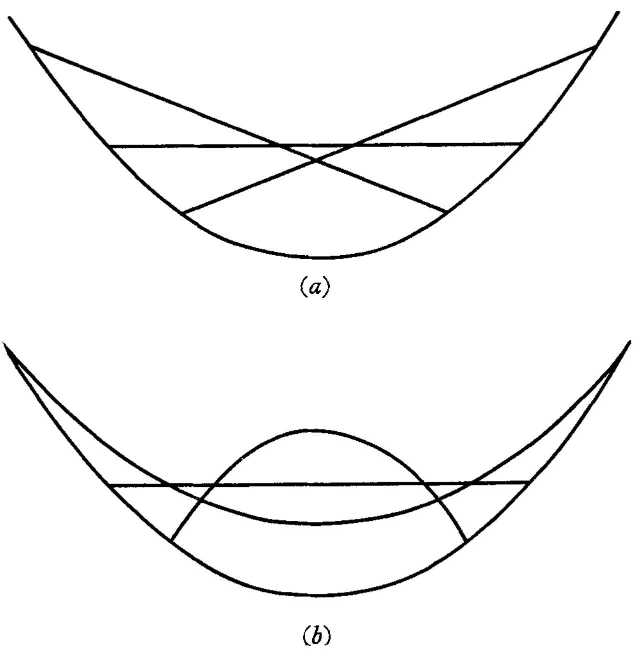

FIGURE 1. Mean and extreme positions of the free surface (vertical scale exaggerated); (*a*) pure displacement, (*b*) pure distension.

and the minimum value of the absolute energy is

$$K_m = E_m + \tfrac{1}{2}fJ = \frac{1}{8I}\left\{\left(\frac{dI}{dt}\right)^2 + \nu^2 I^2\right\} + \frac{J^2}{2I} + \frac{2Q^3 g}{9\pi I}. \tag{44}$$

Returning once more to the general case, we use the preceding results to investigate the energy transfer between the distension and the additional motions. Let us consider the quantity $K - K_m = E - E_m$, i.e. the amount by which the energy exceeds the minimum value consistent with Q, I, dI/dt and J. From equations (22) and (44) we obtain

$$K - K_m = \frac{1}{4I}\left(C - 2J^2 - \frac{8Q^3 g}{9\pi}\right). \tag{45}$$

Now $K = K_m$ only if the motion is a pure distension and in all other cases $K > K_m$. Thus a necessary and sufficient condition for the motion to be a pure distension is

$C = 2J^2 + 8Q^3g/9\pi$; in all other cases C is greater than this value. The amount by which C exceeds its minimum possible value is a measure of the amount of energy in the additional motions. This can be shown formally by considering the asterisked equivalent of equation (44)

$$K_m^* = \frac{\nu^{*2} I_0}{8} + \frac{J^2}{2I_0} + \frac{2Q^3 g}{9\pi I_0}.$$

If we substitute for ν^{*2} from equations (38) and (34) we obtain

$$K_m^* = \frac{\alpha^* g I_0}{2} + \frac{J^2}{I_0} + \frac{2Q^3 g}{9\pi I_0},$$

and from equation (39) and the preceding one we have

$$K^* = \alpha^* g I_0 + J^2/I_0.$$

Returning to equation (45) and substituting for $C - 2J^2$ from equation (35) we find

$$K - K_m = \frac{I_0}{I}\left(\frac{I_0 \alpha^* g}{2} - \frac{2Q^3 g}{9\pi I_0}\right).$$

Finally, using the preceding expressions for K^* and K_m^*, we reduce the energy equation to

$$K - K_m = \frac{I_0}{I}(K^* - K_m^*). \tag{49}$$

This equation† (or equation (45)) completely describes the energy transfer between the distension and the additional motions. All the quantities are constant, except I and K_m, and since I is a function of time so also is K_m. When I increases, the distension energy K_m increases also, and energy is transferred from the additional motions to the distension. When the liquid has spread far enough, which it always will in the collapsing case ($\nu^2 \leqslant 0$), the energy in the additional motions will become negligible.

4. General discussion

We are now in a position to discuss in general terms the behaviour of a shallow mass of liquid of arbitrary shape in an arbitrary state of (shallow-water) motion, lying on a horizontal surface or on a paraboloid of revolution. The motion of the liquid can always be split into three parts:

(i) The motion of the centre of gravity. This is entirely independent of motions relative to it and is controlled by a pair of simple linear equations (equations (5) and (6)). As far as the relative motions are concerned we can regard the centre of gravity as stationary at the origin.

(ii) A simple distension (rotation and dilatation). This motion is influenced only by the *total energy* of the additional motions which acts as a 'pressure' tending to increase the spread of the liquid. If the spread of the liquid does increase, then some of this energy is converted to distension energy. The distension is controlled by a simple linear equation (equation (21)).

† This equation can also be derived more directly by showing that $C^* = C$ and then considering the asterisked equivalent of equation (45).

(iii) The additional motions. The main effects of the distension on these motions are, first, a slowing down or speeding up of every aspect of the motion (whether vortices or gravity waves) according as the liquid as a whole is stretched or contracted, and secondly, a general stabilization, since, when considering the additional motions, the liquid must be regarded as lying on a paraboloid $Z = \frac{1}{2}\alpha^* r^{*2}$, α^* being essentially positive (see equations (39) and (40)).

Let us first consider the simple case of a liquid dome of arbitrary shape, in an arbitrary state of shallow-water motion, lying on a horizontal surface and suppose that the effect of the earth's rotation is negligible ($\alpha = 0, f = 0$). The centre of gravity of the dome will move in a straight line at constant speed; at the same time the dome will collapse so that the moment of inertia about its centre of gravity (from equation (21) with $\nu^2 = 0$) is given by

$$I = 2Kt^2 + I_0, \tag{50}$$

where the origin of t has been chosen so that $dI/dt = 0$ when $t = 0$ and I_0 (used in the definition of the quantities with asterisks) is taken to be the initial value of I. The additional motions must be considered as taking place in a parabolic bowl and they will presumably be oscillatory. However, these motions will *slow down continuously* and it is of interest to investigate the exact form of this time transformation. From equations (27) and (50) we have

$$dt^* = \frac{dt}{1 + 2Kt^2/I_0},$$

so

$$t^*\left(\frac{2K}{I_0}\right)^{\frac{1}{2}} = \tan^{-1}\left[t\left(\frac{2K}{I_0}\right)^{\frac{1}{2}}\right]. \tag{51}$$

The implications of this equation are curious; not only do the additional motions slow down as the liquid spreads out, but even though t increases without limit t^* never exceeds the value

$$T = \frac{\pi}{2}\left(\frac{I_0}{2K}\right)^{\frac{1}{2}}. \tag{52}$$

Thus if the additional motions are oscillatory only a finite number of oscillations can be executed. This is illustrated schematically in figure 2. The value of T for a dome of central depth 1 m and radius 1 km is about 10 min.

When ν^2 is negative, the additional motions behave in much the same way as in the previous case. If we put $\nu^2 = -n^2$, the appropriate solution of equation (21) is

$$I = (I_0 + 4K/n^2)\cosh nt - 4K/n^2, \tag{53}$$

and equation (26) gives

$$dt^* = \frac{I_0\,dt}{(I_0 + 4K/n^2)\cosh nt - 4K/n^2},$$

whence

$$\frac{nt^*}{2}\left(\frac{I_0 + 8K/n^2}{I_0}\right)^{\frac{1}{2}} = \tan^{-1}\left\{\frac{(I_0 + 4K/n^2)\,e^{nt} - 4K/n^2}{[I_0(I_0 + 8K/n^2)]^{\frac{1}{2}}}\right\}, \tag{54}$$

and as t increases without limit, the right-hand side of this equation increases to a limiting value of $\frac{1}{2}\pi$, so that t^* never exceeds $T = \pi[I_0/(n^2 I_0 + 8K)]^{\frac{1}{2}}$ (this reduces to equation (52) when $n = 0$).

To understand the importance of this limiting time, it is necessary to compare it with typical periods of the simpler modes of oscillation as deduced from perturbation theory. The characteristic frequency is

$$\nu^{*2} = 2C/I_0^2 \quad \text{(from equations (38), (34) and (35))}$$
$$= (n^2 I_0 + 8K)/I_0 \quad \text{(from equations (22) and (53))}.$$

The characteristic period τ^* is therefore given by

$$\tau^* = 2\pi/\nu^* = 2T,$$

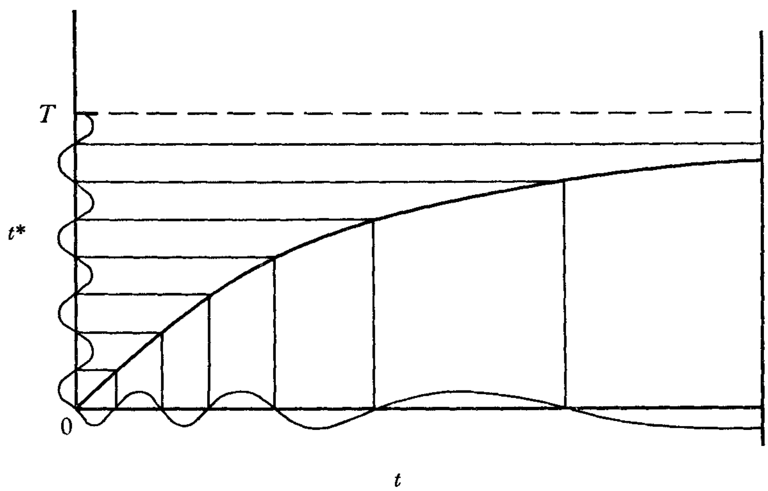

FIGURE 2. The relation between t^* and t, indicating how a regular oscillation in t^* becomes a continuously retarding oscillation in t with only a finite number of cycles.

which shows that, in the case of the simpler modes of the additional oscillations in which most of the energy would usually be concentrated, there would not be time available for the completion of one oscillation.

In the cases just considered, the liquid dome is of necessity collapsing, so there will always be a distension. However, in the oscillatory case, the liquid 'pulsates' with arbitrary amplitude and we can consider the case of zero amplitude when the distension reduces to a constant rotation. Let us then consider, in more detail than heretofore, the equilibrium conditions when the absolute angular momentum, J, and the volume, Q, are given and the liquid is in a state of solid rotation (this incidentally will give the minimum absolute energy compatible with the given values of Q and J). If we denote the moment of inertia in this equilibrium position by I_0, and the absolute energy by K_0, then we have from equation (21)

$$\nu^2 I_0 = 4K_0,$$

and from equation (44)

$$K_0 = \frac{\nu^2 I_0}{8} + \frac{J^2 + 4gQ^3/9\pi}{2I_0},$$

whence

$$I_0^2 = \frac{4(J^2 + 4gQ^3/9\pi)}{\nu^2}, \tag{55}$$

and

$$K_0^2 = \frac{\nu^2 (J^2 + 4gQ^3/9\pi)}{4}. \tag{56}$$

We obtain exactly the same result if we determine the conditions for a minimum absolute energy from equation (44).

If, for given values of Q and J, the absolute energy exceeds K_0, then there will be energy available for displacement, for distension or for additional motions. The displacement motions can be determined easily from the simple linear equations (5) and (6), when the initial position and velocity of the centre of gravity are known. We do not propose to discuss these motions here but will assume that the centre of gravity is stationary at the origin.

Let us suppose that there is a pulsation present of amplitude a. The appropriate solution of equation (21) is

$$I = 4K/\nu^2 + a\cos\nu t. \tag{57}$$

We then find, from equation (22), that

$$C = 8K^2/\nu^2 - \tfrac{1}{2}a^2\nu^2. \tag{58}$$

If we take equation (55) as the definition of I_0 then

$$I_0 = 4K_0/\nu^2$$

and from equation (49)

$$K - K_m = 4K_0(K^* - K_m^*)/I\nu^2. \tag{59}$$

Then by eliminating $K - K_m$ between equations (59) and (45) and substituting for C using equation (58), we find

$$K^2 = K_0^2 + a^2\nu^4/16 + 2K_0(K^* - K_m^*). \tag{60}$$

This shows how the *square* of the absolute energy is distributed in the mean between the minimum energy state, the pulsation and the additional motions. In the particular case where there are no additional motions then $K^* = K_m^*$ and

$$K^2 = K_0^2 + a^2\nu^4/16, \tag{61}$$

and equation (60) can be put in the form

$$I = \frac{4}{\nu^2}\{K + (K^2 - K_0^2)^{\frac{1}{2}}\cos\nu t\}, \tag{62}$$

which indicates clearly how an increase in available energy (with given values of J and Q and no additional motions) causes an increase both in amplitude of pulsation and in mean value of the moment of inertia.

Returning to the more general case, where the additional motions do not vanish, from equations (27) and (57) the time scale of these motions is given by

$$dt^* = \frac{I_0\,dt}{4K/\nu^2 + a\cos\nu t}. \tag{63}$$

It is convenient to introduce a constant I_1, where

$$I_1 = 16K^2/\nu^4 - a^2 = \frac{16K_0}{\nu^4}(K_0 + K^* - K_m^*) \tag{64}$$

(I_1 is, in fact, the constant moment of inertia that the liquid would possess if the energy of the pulsation were reduced to a minimum without changing the energy of the additional motions). Using this notation we find

$$\sin(\nu t^* I_1/I_0) = \frac{I_1 \sin \nu t}{4K/\nu^2 + a\cos\nu t}.$$

We see, therefore, that the pulsation causes an alternate speeding up and slowing down of the additional motions. The mean effect can be found by putting $\nu t = m\pi$, whence

$$t^* = tI_0/I_1, \tag{65}$$

which gives a general slowing down as compared with infinitesimal additional motions superimposed on the minimum energy state (with given J and Q). That this is a general slowing down dependent on the energy of the additional motions can best be seen from the relation

$$(I_0/I_1)^2 = K_0/(K_0 + K^* - K_m^*),$$

which follows immediately from equation (64) and the definition of I_0.

As before the characteristic frequency of the additional motions is

$$\nu^{*2} = 2C/I_0^2,$$

so, from equations (58) and (64), we find

$$\nu^* = \nu I_1/I_0.$$

This increase in frequency is compensated by the slowing down associated with the change in time co-ordinate (equation (65)). This compensation is complete when $J = 0$ because the frequencies of the modes of additional oscillations are then all multiples of ν^*.

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